

# Conley index and stable sets for flows on flag bundles

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## Abstract

Consider a continuous flow of automorphisms of a  $G$ -principal bundle which is chain transitive on its compact Hausdorff base. Here  $G$  is a connected noncompact semi-simple Lie group with finite center. The finest Morse decomposition of the induced flows on the associated flag bundles were obtained in previous articles. Here we describe the stable sets of these Morse components and, under an additional assumption, their Conley indices.

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## 1 Introduction

The subject matter of this paper are the flows on flag bundles. The questions to be treated are the Conley indices and the stable (and unstable) sets of the

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Morse components.

We consider the following setup (that already appears in [5], [17], [22] and elsewhere): Start with a principal bundle  $Q \rightarrow X$  whose structural group  $G$  is semi-simple non compact (or slightly more generally the Lie algebra of  $G$  is reductive and  $G$  satisfies some mild assumptions) and  $X$  is a compact Hausdorff space. The group  $G$  acts freely (on the right) on  $Q$  having  $X$  as the space of orbits. We assume throughout that the bundle is locally trivial. Now, let  $\phi_t$  ( $t \in \mathbb{Z}$  or  $\mathbb{R}$ ) be a flow on  $Q$  which commutes with the right action of  $G$ :  $\phi_t(p \cdot a) = \phi_t(p) \cdot a$ ,  $p \in Q$  and  $a \in G$ , where  $p \cdot a$  denotes the right action. Then  $\phi_t$  induces flows on any associated bundle  $Q \times_G F$  where  $F$  is a space acted on the left by  $G$ . We mainly take  $F$  to be a flag manifold of  $G$ .

This set up includes the linear flows on vector bundles and on the corresponding projective and Grassmann bundles, for which there is an extensive literature (we cite, for instance, Coloniuss-Kliemann [6], Sacker-Sell [19], Salamon-Zehnder [18], and their references). For these bundles one can take  $G = \mathrm{Gl}(n, \mathbb{R})$  for the structure group.

The finest Morse decomposition for the flows on flag bundles were described in [5] and [17] with the assumption that the flow on the base space is chain transitive. From this description it emerges a reduction of  $Q$  to a subbundle  $Q_\phi$  that is invariant under the flow. The structure group of the reduced bundle is the centralizer  $Z_{H_\phi} \subset G$  of a suitable element  $H_\phi$  of the Lie algebra  $\mathfrak{g}$  of  $G$ . When  $G = \mathrm{Gl}(n, \mathbb{R})$  this centralizer is a subgroup of block diagonal matrices, which suggested the name of block reduction of the flow (see [22], Section 5, and Section 5 below).

The block reduction is one of our main tools. We use it to transport to flag bundles constructions made on the flag manifolds. In fact, by the very definition of  $Q_\phi$  in [22] the union of Morse components in a flag bundle is the set  $\{q \cdot \mathrm{fix}_{H_\phi} : q \in Q_\phi\}$  where  $\mathrm{fix}_{H_\phi}$  is the set of fixed points of the action on a flag manifold of  $\exp H_\phi$  (below we use a different notation, that distinguish the Morse components as well as the specific flag manifold). In this paper we make a similar construction for the stable and unstable sets of the Morse components. They are obtained by plugging into the fibers of a flag bundle the stable sets on the flag manifolds (see Theorem 5.3).

The stable and unstable sets are used to describe the homotopy (and hence the cohomology) Conley index of a Morse component  $\mathcal{M}$ . It turns out that the homotopy index is the Thom space of a vector bundle  $\mathcal{U}$  over  $\mathcal{M}$  (see Theorem 7.4; we recall the definition of the Thom space in Section 2). The

fibers of  $\mathcal{U}$  are the tangent spaces (along the the fibers of the flag bundle) to the unstable set of  $\mathcal{M}$ . By Thom isomorphism the (co)homology Conley index is the displaced (co)homology of  $\mathcal{M}$ . In some cases (e.g. the base space is contractible) the (co)homology of  $\mathcal{M}$  reduces to the (co)homology of a flag manifold (see [14] and [3]).

To prove Theorem 7.4 we first linearize the flow around  $\mathcal{M}$  by building a conjugation between a linear flow on  $\mathcal{U}$  and the flow restricted to a neighborhood of  $\mathcal{M}$ .

However, in order that our construction of the linearization become well defined we make an additional assumption about the flow. Namely, that the block reduction subbundle  $Q_\phi$  admits a further  $\phi_t$ -invariant reduction to a subbundle  $C_\phi$  whose structural group  $L_{H_\phi}$  is a subgroup of  $Z_{H_\phi}$ . The subgroup  $L_{H_\phi}$  is the direct product of a compact group by an abelian vector group. If  $G = \text{Gl}(n, \mathbb{R})$  then  $L_{H_\phi}$  a subgroup of block diagonal matrices where each block is a conformal matrix (product of an orthogonal matrix by a scalar one). This suggests to call  $C_\phi$  a block conformal reduction of the flow.

In general such a conformal reduction may not exist. Although we do not know the full significance of this geometric assumption there are some clues in the literature about its reliability. At this regard we mention the results on Jordan decomposition of cocycles by Arnold-Cong-Oseledets [1], where block conformal matrices are produced by the support of invariant measures on projective and sphere bundles. On the other hand our  $Z_{H_\phi}$ -reduction  $Q_\phi$  is produced by the chain recurrent components on flag bundles. Hence one may suspect that the reason for the existence of a conformal reduction  $C_\phi$  stays in the relation between the supports of invariant measures and the chain components.

## 2 Preliminaries

Troughout the article we will use the notation of principal and associated bundles of Kobayashi-Nomizu [15]. In what follows we establish some other notations and preliminar results.

Let  $\pi : Q \rightarrow X$  be a principal bundle with structural group  $G$  and compact Hausdorff base space  $X$ . Let  $\phi_t$  be a right invariant flow on  $Q$ . We assume throughout that the induced flow on  $X$  is chain transitive.

## 2.1 Flows on topological spaces

Let  $\phi : \mathbb{T} \times E \rightarrow E$  be a continuous flow on a compact Hausdorff space  $E$ , with discrete  $\mathbb{T} = \mathbb{Z}$  or continuous  $\mathbb{T} = \mathbb{R}$  time. Fix an invariant set  $\mathcal{M} \subset E$ . We define its stable and unstable sets respectively as

$$\text{st}(\mathcal{M}) = \{x \in E : \omega(x) \subset \mathcal{M}\}, \quad \text{un}(\mathcal{M}) = \{x \in E : \omega^*(x) \subset \mathcal{M}\}.$$

and its attractor and repeller domains respectively as

$$\mathcal{A}(\mathcal{M}) = \{x \in E : \Omega(x) \subset \mathcal{M}\}, \quad \mathcal{R}(\mathcal{M}) = \{x \in E : \Omega^*(x) \subset \mathcal{M}\},$$

where  $\omega(x)$ ,  $\omega^*(x)$  are the limit sets of  $x$ , and  $\Omega(x)$ ,  $\Omega^*(x)$  are the chain limit sets of  $x$ . Since we have that  $\omega(x) \subset \Omega(x)$ , it follows that  $\text{st}(\mathcal{M}) \subset \mathcal{A}(\mathcal{M})$  and analogously  $\text{un}(\mathcal{M}) \subset \mathcal{R}(\mathcal{M})$ . For the concepts of Morse decompositions and its relations with chain transitivity see [7], [6], [16]. If  $\{\mathcal{M}_i\}_{i \in I}$  is a Morse decomposition of  $E$  then  $E$  decomposes as the disjoint union of stable sets  $\text{st}(\mathcal{M}_i)$ .

We now recall the definition of the Conley index. We note that, when working with the Conley index, we restrict ourselves to the case of a continuous-time flow. A neighborhood  $U \subset E$  of the invariant set  $\mathcal{M}$  is isolating if  $\mathcal{M}$  is the maximal invariant set inside  $U$ . A pair  $(N_1, N_0)$  of subsets of  $E$  with  $N_1 \supset N_0$  is an index pair for  $\mathcal{M}$  if it satisfies the following three conditions

- 1)  $\text{cl}(N_1 \setminus N_0)$  is an isolating neighborhood of  $\mathcal{M}$ .
- 2)  $N_0$  is positively invariant with respect to  $N_1$ , i.e., if  $x \in N_0$  and  $\phi_t(x) \in N_1$ , for some  $t \geq 0$ , then  $\phi_t(x) \in N_0$ .
- 3) If  $x \in N_1$  is such that there exists  $t' > 0$  with  $\phi_{t'}(x) \notin N_1$ , then there exists  $t_m \geq 0$  such that  $\phi_{t_m}(x) \in N_0$  and  $\phi_t(x) \in N_1$ , for every  $t \in [0, t_m]$ .

The Conley index of  $\mathcal{M}$  is defined as the pointed homotopy class  $h(\mathcal{M})$  of the quotient space  $N_1/N_0$  with basepoint  $[N_0]$ . A fundamental result of the abstract theory (see Conley-Zehnder [8]) is that index pairs for an isolated invariant set  $\mathcal{M}$  always exist and that the Conley index of  $\mathcal{M}$  is independent of a chosen index pair.

Let  $H^*$  denote Čech cohomology with coefficients in some fixed ring. The cohomological Conley index of  $\mathcal{M}$  is the reduced cohomology of the Conley index with respect to its basepoint and is denoted by

$$CH^*(\mathcal{M}) = \tilde{H}^*(h(\mathcal{M})).$$

We now recall a construction due to Thom that will be used in the computation of our indexes. Let  $p : \mathcal{V} \rightarrow B$  be an  $n$ -dimensional vector bundle over a paracompact Hausdorff base  $B$ . The Thom space  $T(\mathcal{V})$  of  $\mathcal{V}$  is the pointed homotopy class of the quotient space  $\mathcal{V}/(\mathcal{V} - 0)$  with basepoint  $[(\mathcal{V} - 0)]$ . A Thom class  $U$  for  $\mathcal{V}$  is an element  $U \in H^n(\mathcal{V}, \mathcal{V} - 0)$  such that its restriction to  $H^n(\mathcal{V}_b, (\mathcal{V} - 0)_b)$  is a generator, for all  $b \in B$ . A Thom class for  $\mathbb{Z}_2$  coefficients always exists, and it exists for  $\mathbb{Z}$  coefficients if and only if  $\mathcal{V}$  is orientable (see e.g. [10], Theorem 4D.10). If there exists a Thom class  $U$  then the map

$$H^*(B) \rightarrow H^{*+n}(\mathcal{V}, \mathcal{V} - 0), \quad \alpha \mapsto p^*(\alpha) \cup U.$$

is Thom isomorphism (see [10], Corollary 4D.9). Here  $\cup$  is the cup product and  $p^*$  is the induced map in cohomology of the projection  $p$ . Passing from relative to reduced cohomology the Thom isomorphism becomes

$$H^*(B) \rightarrow \tilde{H}^{*+n}(T(\mathcal{V})), \quad \alpha \mapsto p^*(\alpha) \cup U.$$

Put a riemannian metric  $|\cdot|$  in  $\mathcal{V}$ , let  $D(\mathcal{V}) = \{v \in \mathcal{V} : |v| \leq 1\}$  be the disk bundle and let  $S(\mathcal{V}) = \{v \in \mathcal{V} : |v| = 1\}$  be the sphere bundle of  $\mathcal{V}$ . The inclusion of pairs  $(D\mathcal{V}, S\mathcal{V}) \subset (\mathcal{V}, \mathcal{V} - 0)$  is easily seen to induce a pointed homotopy equivalence between  $D(\mathcal{V})/S(\mathcal{V})$  with basepoint  $[S(\mathcal{V})]$  and the Thom space  $T(\mathcal{V})$ , so the Thom space of  $\mathcal{V}$  can alternatively be defined as the pointed homotopy type this quotient.

Now we recall the Morse equation obtained in [8]. Let  $(Y_1, Y_0)$  be a pair of compact Hausdorff spaces with  $Y_1 \supset Y_0$  and denote by  $H(Y_1, Y_0)$  its Čech cohomology with coefficients in some fixed ring. Assuming each module  $H^j(Y_1, Y_0)$  to be of finite rank we define the Poincaré polynomial of the pair  $(Y_1, Y_0)$  by the formal sum

$$P(t, Y_1, Y_0) := \sum_{j \geq 0} t^j \text{rank } H^j(Y_1, Y_0).$$

We define the Poincaré polynomial of the space  $Y$  by  $P(t, Y) := P(t, Y, \emptyset)$  and the Poincaré polynomial of the index of an isolated invariant set  $\mathcal{M} \subset E$

by

$$CP(t, \mathcal{M}) = P(t, N_1, N_0),$$

where  $(N_1, N_0)$  is an index pair for  $\mathcal{M}$ . Since for Čech cohomology we have  $H(N_1, N_0) = \tilde{H}(N_1/N_0) = CH(\mathcal{M})$ , it follows that

$$CP(t, \mathcal{M}) := \sum_{j \geq 0} t^j \text{rank } CH^j(\mathcal{M}),$$

so that this polynomial does not depend on the chosen index pair. The Morse equation relates the cohomology of the indices with the cohomology of the whole space  $E$ .

**Theorem 2.1** *Let  $\{\mathcal{M}_i\}_{i \in I}$  be a Morse decomposition of  $E$ , then*

$$\sum_{i \in I} CP(t, \mathcal{M}_i) = P(t, E) + (1 + t)R(t).$$

*If the coefficient ring is  $\mathbb{Z}$  then the coefficients of  $R(t)$  are non-negative.*

## 2.2 Semi-simple Lie Theory

For the theory of semi-simple Lie groups and their flag manifolds we refer to Duistermaat-Kolk-Varadarajan [9], Helgason [11], Knapp [13] and Warner [25]. To set notation let  $G$  be a connected noncompact semi-simple Lie group with Lie algebra  $\mathfrak{g}$ . We assume throughout that  $G$  has finite center. Fix a Cartan involution  $\theta$  of  $\mathfrak{g}$  with Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ . The form  $B_\theta(X, Y) = -\langle X, \theta Y \rangle$ , where  $\langle \cdot, \cdot \rangle$  is the Cartan-Killing form of  $\mathfrak{g}$ , is an inner product.

Fix a maximal abelian subspace  $\mathfrak{a} \subset \mathfrak{s}$  and a Weyl chamber  $\mathfrak{a}^+ \subset \mathfrak{a}$ . We let  $\Pi$  be the set of roots of  $\mathfrak{a}$ ,  $\Pi^+$  the positive roots corresponding to  $\mathfrak{a}^+$ ,  $\Sigma$  the set of simple roots in  $\Pi^+$  and  $\Pi^- = -\Pi^+$  the negative roots. For a root  $\alpha \in \Pi$  we denote by  $H_\alpha \in \mathfrak{a}$  its coroot so that  $B_\theta(H_\alpha, H) = \alpha(H)$  for all  $H \in \mathfrak{a}$ . The Iwasawa decomposition of the Lie algebra  $\mathfrak{g}$  reads  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^\pm$  with  $\mathfrak{n}^\pm = \sum_{\alpha \in \Pi^\pm} \mathfrak{g}_\alpha$  where  $\mathfrak{g}_\alpha$  is the root space associated to  $\alpha$ . As to the global decompositions of the group we write  $G = KS$  and  $G = KAN^\pm$  with  $K = \exp \mathfrak{k}$ ,  $S = \exp \mathfrak{s}$ ,  $A = \exp \mathfrak{a}$  and  $N^\pm = \exp \mathfrak{n}^\pm$ .

The Weyl group  $\mathcal{W}$  associated to  $\mathfrak{a}$  is the finite group generated by the reflections over the root hyperplanes  $\alpha = 0$  in  $\mathfrak{a}$ ,  $\alpha \in \Pi$ .  $\mathcal{W}$  acts on  $\mathfrak{a}$  by isometries and can be alternatively be given as  $\mathcal{W} = M^*/M$  where  $M^*$  and

$M$  are the normalizer and the centralizer of  $A$  in  $K$ , respectively. We write  $\mathfrak{m}$  for the Lie algebra of  $M$ . There is a unique element  $w^- \in \mathcal{W}$  which takes the simple roots  $\Sigma$  to  $-\Sigma$ ,  $w^-$  is called the principal involution of  $\mathcal{W}$ . The Bruhat-Chevalley order in  $\mathcal{W}$  is a partial order given as follows. Take for  $w \in \mathcal{W}$  a reduced expression  $w = s_1 \cdots s_n$  as a product of reflections with respect to the simple roots  $\Sigma$ . Then  $\bar{w} \leq w$  if and only if there are integers  $1 \leq i_1 \leq \cdots \leq i_j \leq n$  such that  $\bar{w} = s_{i_1} \cdots s_{i_j}$  is a reduced expression for  $\bar{w}$ . We have that  $w^-$  is the greatest element of  $\mathcal{W}$  in this order.

Associated to a subset of simple roots  $\Theta \subset \Sigma$  there are several Lie algebras and groups (cf. [25], Section 1.2.4): We write  $\mathfrak{g}(\Theta)$  for the (semi-simple) Lie subalgebra generated by  $\mathfrak{g}_\alpha$ ,  $\alpha \in \Theta$ , and put  $\mathfrak{k}(\Theta) = \mathfrak{g}(\Theta) \cap \mathfrak{k}$ ,  $\mathfrak{a}(\Theta) = \mathfrak{g}(\Theta) \cap \mathfrak{a}$ , and  $\mathfrak{n}^\pm(\Theta) = \mathfrak{g}(\Theta) \cap \mathfrak{n}^\pm$ . The simple roots of  $\mathfrak{g}(\Theta)$  are given by  $\Theta$ , more precisely, by restricting the functionals of  $\Theta$  to  $\mathfrak{a}(\Theta)$ . The coroots  $H_\alpha$ ,  $\alpha \in \Theta$ , form a basis for  $\mathfrak{a}(\Theta)$ . Let  $G(\Theta)$  and  $K(\Theta)$  be the connected groups with Lie algebra  $\mathfrak{g}(\Theta)$  and  $\mathfrak{k}(\Theta)$ , respectively. Then  $G(\Theta)$  is a connected semi-simple Lie group with finite center. Let  $A(\Theta) = \exp \mathfrak{a}(\Theta)$ ,  $N^\pm(\Theta) = \exp \mathfrak{n}^\pm(\Theta)$ . We have the Iwasawa decomposition  $G(\Theta) = K(\Theta)A(\Theta)N^\pm(\Theta)$ . Let  $\mathfrak{a}_\Theta = \{H \in \mathfrak{a} : \alpha(H) = 0, \alpha \in \Theta\}$  be the orthocomplement of  $\mathfrak{a}(\Theta)$  in  $\mathfrak{a}$  with respect to the  $B_\theta$ -inner product and put  $A_\Theta = \exp \mathfrak{a}_\Theta$ . The subset  $\Theta$  singles out the subgroup  $\mathcal{W}_\Theta$  of the Weyl group which acts trivially on  $\mathfrak{a}_\Theta$ . Alternatively  $\mathcal{W}_\Theta$  can be given as the subgroup generated by the reflections with respect to the roots  $\alpha \in \Theta$ . The restriction of  $w \in \mathcal{W}_\Theta$  to  $\mathfrak{a}(\Theta)$  furnishes an isomorphism between  $\mathcal{W}_\Theta$  and the Weyl group  $\mathcal{W}(\Theta)$  of  $G(\Theta)$ .

The standard parabolic subalgebra of type  $\Theta \subset \Sigma$  with respect to chamber  $\mathfrak{a}^+$  is defined by

$$\mathfrak{p}_\Theta = \mathfrak{n}^-(\Theta) \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+.$$

The corresponding standard parabolic subgroup  $P_\Theta$  is the normalizer of  $\mathfrak{p}_\Theta$  in  $G$ . It has the Iwasawa decomposition  $P_\Theta = K_\Theta AN^+$ . The empty set  $\Theta = \emptyset$  gives the minimal parabolic subalgebra  $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$  whose minimal parabolic subgroup  $P = P_\emptyset$  has Iwasawa decomposition  $P = MAN^+$ . Let  $\Delta \subset \Theta \subset \Sigma$ . Then  $P_\Delta \subset P_\Theta$ , also we denote by  $P(\Theta)_\Delta$  the parabolic subgroup of  $G(\Theta)$  of type  $\Delta$ .

We let  $Z_\Theta$  be the centralizer of  $\mathfrak{a}_\Theta$  in  $G$  and  $K_\Theta = Z_\Theta \cap K$ . We have that  $K_\Theta$  decomposes as  $K_\Theta = MK(\Theta)$  and that  $Z_\Theta$  decomposes as  $Z_\Theta = MG(\Theta)A_\Theta$  which implies that  $Z_\Theta = K_\Theta AN(\Theta)$  is an Iwasawa decomposition of  $Z_\Theta$  (which is a reductive Lie group). Let  $\Delta \subset \Sigma$ , then<sup>1</sup>  $\mathfrak{a}_{\Theta \cap \Delta} = \mathfrak{a}_\Theta + \mathfrak{a}_\Delta$ .

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<sup>1</sup>Using that  $\mathfrak{a}_\Theta = \mathfrak{a}(\Theta)^\perp$  and that  $\mathfrak{a}(\Theta \cap \Delta) = \mathfrak{a}(\Theta) \cap \mathfrak{a}(\Delta)$  this follows by taking perp

Thus it follows that  $Z_{\Theta \cap \Delta} = Z_{\Theta} \cap Z_{\Delta}$ ,  $K_{\Theta \cap \Delta} = K_{\Theta} \cap K_{\Delta}$  and  $P_{\Theta \cap \Delta} = P_{\Theta} \cap P_{\Delta}$ . For  $H \in \mathfrak{a}$  we denote by  $Z_H$ ,  $\mathcal{W}_H$  etc. the centralizer of  $H$ , respectively, in  $G$ ,  $\mathcal{W}$  etc., except when explicitly noted. When  $H \in \text{cl}\mathfrak{a}^+$  we put

$$\Theta(H) = \{\alpha \in \Sigma : \alpha(H) = 0\},$$

and we have  $Z_H = Z_{\Theta(H)}$ ,  $K_H = K_{\Theta(H)}$ ,  $N_H^+ = N^+(\Theta(H))$  and  $\mathcal{W}_H = \mathcal{W}_{\Theta(H)}$ .

Let  $\mathfrak{n}_{\Theta}^{\pm} = \sum_{\alpha \in \Pi^{\pm} - \langle \Theta \rangle} \mathfrak{g}_{\alpha}$  and  $N_{\Theta}^{\pm} = \exp(\mathfrak{n}_{\Theta}^{\pm})$ . Then  $N^{\pm}$  decomposes as  $N^{\pm} = N(\Theta)^{\pm} N_{\Theta}^{\pm}$  where  $N(\Theta)^{\pm}$  normalizes  $N_{\Theta}^{\pm}$  and  $N(\Theta)^{\pm} \cap N_{\Theta}^{\pm} = 1$ . We have that  $\mathfrak{g} = \mathfrak{n}_{\Theta}^{-} \oplus \mathfrak{p}_{\Theta}$ , that  $N_{\Theta}^{-} \cap P_{\Theta} = 1$  and also that  $P_{\Theta}$  is the normalizer of  $\mathfrak{n}_{\Theta}^{+}$  in  $G$ .  $P_{\Theta}$  decomposes as  $P_{\Theta} = Z_{\Theta} N_{\Theta}^{+}$ , where  $Z_{\Theta}$  normalizes  $N_{\Theta}^{+}$  and  $Z_{\Theta} \cap N_{\Theta}^{+} = 1$ . We write  $\mathfrak{p}_{\Theta}^{-} = \theta(\mathfrak{p}_{\Theta})$  for the parabolic subalgebra opposed to  $\mathfrak{p}_{\Theta}$ . It is conjugate to the parabolic subalgebra  $\mathfrak{p}_{\Theta^*}$  where  $\Theta^* = -(w^{-})\Theta$  is the dual to  $\Theta$  and  $w^{-}$  is the principal involution of  $\mathcal{W}$ . More precisely,  $\mathfrak{p}_{\Theta}^{-} = k\mathfrak{p}_{\Theta^*}$  where  $k \in M^*$  is a representative of  $w^{-}$ . If  $P_{\Theta}^{-}$  is the parabolic subgroup associated to  $\mathfrak{p}_{\Theta}^{-}$  then  $Z_{\Theta} = P_{\Theta} \cap P_{\Theta}^{-}$  and  $P_{\Theta}^{-} = Z_{\Theta} N_{\Theta}^{-}$ , where  $Z_{\Theta}$  normalizes  $N_{\Theta}^{-}$  and  $Z_{\Theta} \cap N_{\Theta}^{-} = 1$ .

The flag manifold of type  $\Theta$  is the orbit  $\mathbb{F}_{\Theta} = \text{Ad}(G)\mathfrak{p}_{\Theta}$  with base point  $b_{\Theta} = \mathfrak{p}_{\Theta}$ , which identifies with the homogeneous space  $G/P_{\Theta}$ . Since the center of  $G$  normalizes  $\mathfrak{p}_{\Theta}$ , the flag manifold depends only on the Lie algebra  $\mathfrak{g}$  of  $G$ . The empty set  $\Theta = \emptyset$  gives the maximal flag manifold  $\mathbb{F} = \mathbb{F}_{\emptyset}$  with basepoint  $b = b_{\emptyset}$ . If  $\Delta \subset \Theta$  then there is a  $G$ -equivariant projection  $\mathbb{F}_{\Delta} \rightarrow \mathbb{F}_{\Theta}$  given by  $gb_{\Delta} \mapsto gb_{\Theta}$ ,  $g \in G$ .

The above subalgebras of  $\mathfrak{g}$ , which are defined by the choice of a Weyl chamber of  $\mathfrak{a}$  and a subset of the associated simple roots, can be defined alternatively by the choice of an element  $H \in \mathfrak{a}$  as follows. First note that the eigenspaces of  $\text{ad}(H)$  in  $\mathfrak{g}$  are the weight spaces  $\mathfrak{g}_{\alpha}$ , and that the centralizer of  $H$  in  $\mathfrak{g}$  is given by  $\mathfrak{z}_H = \sum \{\mathfrak{g}_{\alpha} : \alpha(H) = 0\}$ , where the sum is taken over  $\alpha \in \mathfrak{a}^*$ . Now define the negative and positive nilpotent subalgebras of type  $H$  given by

$$\mathfrak{n}_H^{-} = \sum \{\mathfrak{g}_{\alpha} : \alpha(H) < 0\}, \quad \mathfrak{n}_H^{+} = \sum \{\mathfrak{g}_{\alpha} : \alpha(H) > 0\},$$

and the parabolic subalgebra of type  $H$  which is given by

$$\mathfrak{p}_H = \sum \{\mathfrak{g}_{\alpha} : \alpha(H) \geq 0\}.$$

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on both sides and using that  $(V + W)^{\perp} = V^{\perp} \cap W^{\perp}$  for  $V, W$  linear subspaces.



Denote by  $N_H^\pm = \exp(\mathfrak{n}_H^\pm)$  and by  $P_H$  the normalizer in  $G$  of  $\mathfrak{p}_H$ . Note that  $\mathfrak{n}_H^\pm$ ,  $\mathfrak{p}_H$ ,  $N_H^\pm$  and  $P_H$  are not centralizers of  $H$ : these are the only exceptions for the centralizer notation introduced above. We have clearly that

$$\mathfrak{g} = \mathfrak{n}_H^- \oplus \mathfrak{z}_H \oplus \mathfrak{n}_H^+ \quad \text{and} \quad \mathfrak{p}_H = \mathfrak{z}_H \oplus \mathfrak{n}_H^+.$$

Define the flag manifold of type  $H$  given by the orbit

$$\mathbb{F}_H = \text{Ad}(G)\mathfrak{p}_H.$$

Now choose a chamber  $\mathfrak{a}^+$  of  $\mathfrak{a}$  which contains  $H$  in its closure, consider the simple roots  $\Sigma$  associated to  $\mathfrak{a}^+$  and take  $\Theta(H) \subset \Sigma$ . Since a root  $\alpha \in \Theta(H)$  if, and only if,  $\alpha|_{\mathfrak{a}_{\Theta(H)}} = 0$ , we have that

$$\mathfrak{z}_H = \mathfrak{z}_{\Theta(H)}, \quad \mathfrak{n}_H^\pm = \mathfrak{n}_{\Theta(H)}^\pm, \quad \mathfrak{p}_H = \mathfrak{p}_{\Theta(H)}.$$

So it follows that

$$\mathbb{F}_H = \mathbb{F}_{\Theta(H)},$$

and that the isotropy of  $G$  in  $\mathfrak{p}_H$  is

$$P_H = P_{\Theta(H)} = K_{\Theta(H)}AN^+ = K_HAN^+,$$

since  $K_{\Theta(H)} = K_H$ . We note that we can proceed reciprocally. That is, if  $\mathfrak{a}^+$  and  $\Theta$  are given, we can choose an  $H \in \text{cl}\mathfrak{a}^+$  such that  $\Theta(H) = \Theta$  and describe the objects that depend on  $\mathfrak{a}^+$  and  $\Theta$  by  $H$  (clearly, such an  $H$  is not unique.) We remark that the map

$$\mathbb{F}_H \rightarrow \mathfrak{s}, \quad k\mathfrak{p}_H \mapsto \text{Ad}(k)H, \quad \text{where } k \in K, \quad (1)$$

gives an embedding of  $\mathbb{F}_H$  in  $\mathfrak{s}$  (see Proposition 2.1 of [9]). In fact, the isotropy of  $K$  at  $H$  is  $K_H = K_{\Theta(H)}$  which is, by the above comments, the isotropy of  $K$  at  $\mathfrak{p}_H$ . Define the negative parabolic subalgebra of type  $H$  by

$$\mathfrak{p}_H^- = \sum \{\mathfrak{g}_\alpha : \alpha(H) \leq 0\}$$

and denote by  $P_H^-$  its normalizer in  $G$ . Then we have that  $P_H^- = P_{\Theta(H)}^-$ .

### 3 Gradient flows on flag manifolds

The main result of this section is the construction in Theorem 3.2 of equivariant linearizations for the gradient flows on the flag manifolds induced by the split elements in  $\mathfrak{g}$ . In the next section we plug this linearization on the fibers of the flag bundles to get our main tool in the determination of the Conley indices.

### 3.1 Fixed points, stable and unstable sets

An split element  $H \in \text{cl}\mathfrak{a}^+$  induces a vector field  $\tilde{H}$  on a flag manifold  $\mathbb{F}_\Theta$  with flow  $\exp(tH)$ . This is a gradient vector field with respect to a given Riemannian metric on  $\mathbb{F}_\Theta$  (see [9], Section 3). The connected sets of fixed point of this flow are given by

$$\text{fix}_\Theta(H, w) = Z_H w b_\Theta = K_H w b_\Theta,$$

so that they are in bijection with the cosets in  $\mathcal{W}_H \backslash \mathcal{W} / \mathcal{W}_\Theta$ . Each  $w$ -fixed point connected set has stable manifold given by

$$\text{st}_\Theta(H, w) = N_H^- \text{fix}_\Theta(H, w) = P_H^- w b_\Theta,$$

whose union gives the Bruhat decomposition of  $\mathbb{F}_\Theta$ :

$$\mathbb{F}_\Theta = \coprod_{\mathcal{W}_H \backslash \mathcal{W} / \mathcal{W}_\Theta} \text{st}_\Theta(H, w) = \coprod_{\mathcal{W}_H \backslash \mathcal{W} / \mathcal{W}_\Theta} P_H^- w b_\Theta.$$

The unstable manifold is

$$\text{un}_\Theta(H, w) = N_H^+ \text{fix}_\Theta(H, w) = P_H w b_\Theta.$$

Since the centralizer  $Z_H$  of  $H$  leaves  $\text{fix}_\Theta(H, w)$  invariant and normalizes both  $N_H^-$  and  $N_H^+$ , it follows  $\text{st}_\Theta(H, w)$  and  $\text{un}_\Theta(H, w)$  are  $Z_H$ -invariant. We note that these fixed points and (un)stable sets remain the same if  $H$  is replaced by some  $H' \in \text{cl}\mathfrak{a}^+$  such that  $\Theta(H') = \Theta(H)$ .

For  $X \in \mathfrak{g}$  we denote with  $\tilde{X}$  the vector field induced by  $X$  on  $\mathbb{F}_\Theta$ , and for a subset  $\mathfrak{l} \subset \mathfrak{g}$  and  $x \in \mathbb{F}_\Theta$  we put

$$\mathfrak{l} \cdot x = \{\tilde{X}(x) \in T_x \mathbb{F}_\Theta : X \in \mathfrak{l}\}.$$

With this notation, for some  $x \in \text{fix}_\Theta(H, w)$ , the tangent space to  $\text{fix}_\Theta(H, w)$  at  $x$  is given by  $\mathfrak{z}_H \cdot x$ , while the tangent spaces at  $x$  of the unstable and stable manifolds is given, respectively, by  $\mathfrak{n}_H^\pm \cdot x$ . These tangent spaces form the stable and unstable vector bundles

$$V_\Theta^+(H, w) \rightarrow \text{fix}_\Theta(H, w) \quad \text{and} \quad V_\Theta^-(H, w) \rightarrow \text{fix}_\Theta(H, w)$$

with fibers  $\mathfrak{n}_H^\pm \cdot x$ ,  $x \in \text{fix}_\Theta(H, w)$ , respectively. Their Whitney sum

$$V_\Theta(H, w) = V_\Theta^+(H, w) \oplus V_\Theta^-(H, w) \tag{2}$$

is a normal bundle of  $\text{fix}_\Theta(H, w)$  in  $\mathbb{F}_\Theta$ . Note that thanks to the well known translation formula for vector fields

$$dg_x(\tilde{X}(x)) = (\text{Ad}(g)X)^\sim(gx),$$

for  $g \in G$ , it follows that the bundles  $V_\Theta^\pm(H, w)$  are  $Z_H$ -invariant.

### 3.2 Linearization

In the sequel a special role is played by the subgroup  $L_H$  of  $Z_H$ ,  $H \in \text{cla}^+$ , defined by

$$L_H = K_H A_{\Theta(H)} \subset Z_H. \quad (3)$$

Note that  $L_H$  is the direct product of a compact group  $K_H$  and a vector group  $A_{\Theta(H)}$ , hence we call it the conformal part of  $Z_H$ . In case  $H$  is regular  $L_H = Z_H = MA$ . Also, by the structure of the parabolic subgroups we have

$$\text{fix}_\Theta(H, w) = L_H w b_\Theta \quad \text{and} \quad \text{st}_\Theta(H, w) = L_H N_H^- w b_\Theta.$$

Now we start the main construction of this section, namely of a mapping conjugating the flow  $\exp(tH)$  around a fixed point set  $\text{fix}_\Theta(H, w)$  with a linear flow on the above normal bundle of  $\text{fix}_\Theta(H, w)$ . The existence of such a conjugation is a consequence of general theory on Morse-Bott theory (see [9], Corollary 1.5). However we intend to carry this construction to flag bundles, so we require the conjugation map to be equivariant with respect to  $L_H$ . Unfortunately our method does not yield, in general, equivariance with respect to the whole  $Z_H$ , as desirable (see the next remark). Below we discuss some cases where  $Z_H$ -equivariance holds.

The strategy to get a conjugation consists in defining for each  $x \in \text{fix}_\Theta(H, w)$  a subspace

$$\mathfrak{l}_x \subset \mathfrak{n}_H^- \oplus \mathfrak{n}_H^+$$

satisfying the following requirements:

1.  $\mathfrak{l}_x \cdot x = V_\Theta(H, w)_x = (\mathfrak{n}_H^- \oplus \mathfrak{n}_H^+) \cdot x$ .
2. The map  $X \in \mathfrak{l}_x \mapsto \tilde{X}(x) \in V_\Theta(H, w)$  is injective, and hence a bijection.
3. For every  $g \in L_H$ ,  $\text{Ad}(g)\mathfrak{l}_x = \mathfrak{l}_{gx}$ . (This is to ensure equivariance.)

Once we have the subspaces  $\mathfrak{l}_x$  it is easy to get a conjugation by putting

$$\psi : V_\Theta(H, w) \rightarrow \mathbb{F}_\Theta, \quad \psi(v) = \exp(X)x$$

where  $X \in \mathfrak{l}_x$  is such that  $v = \tilde{X}(x)$ . By the third condition and the translation formula for vector fields, it follows that  $\psi$  is  $L_H$ -equivariant, that is

$$\psi \circ dg = g \circ \psi, \quad g \in L_H.$$

We define the subspaces  $\mathfrak{l}_x$ ,  $x \in \text{fix}_\Theta(H, w)$ , by

$$\mathfrak{l}_x = \text{Ad}(g)\mathfrak{l}_{wb_\Theta}, \quad \mathfrak{l}_{wb_\Theta} = w\mathfrak{n}_\Theta^- \cap (\mathfrak{n}_H^- \oplus \mathfrak{n}_H^+),$$

where  $x = gwb_\Theta$ , with  $g \in L_H$ . In order to show that  $\mathfrak{l}_x$  is well defined, we only have to check that the isotropy of  $wb_\Theta$  in  $L_H$  normalizes  $w\mathfrak{n}_\Theta^-$ , since  $L_H$  already normalizes  $\mathfrak{n}_H^- \oplus \mathfrak{n}_H^+$ .

**Lemma 3.1** *The isotropy of  $wb_\Theta$  in  $L_H$  normalizes  $w\mathfrak{n}_\Theta^-$ .*

**Proof:** Clearly, this isotropy is given by  $P_\Theta^w \cap L_H$ , where the superscript  $w$  denotes conjugation by some representative of  $w$  in  $M^*$ . Using the Iwasawa decompositions

$$P_\Theta^w = K_\Theta^w AN^w, \quad Z_\Theta^w = K_\Theta^w AN(\Theta)^w$$

and the uniqueness of the Iwasawa decomposition  $G = KAN^w$ , we have that

$$P_\Theta^w \cap L_H = (K_\Theta^w \cap K_H)A_{\Theta(H)} \subset Z_\Theta^w,$$

which normalizes  $w\mathfrak{n}_\Theta^-$ . □

In order to verify conditions (1) and (2) stated above, we only need to check that the isotropy subalgebra of  $\mathfrak{g}$  at  $x$  is complemented by  $\mathfrak{l}_x$ . In fact, the isotropy subalgebra at  $x = gwb_\Theta$ ,  $g \in L_H$ , is given by  $\text{Ad}(g)w\mathfrak{p}_\Theta$ . It is clearly complemented in  $\mathfrak{g}$  by  $\mathfrak{l}_x$ , since  $\mathfrak{g} = \mathfrak{p}_\Theta \oplus \mathfrak{n}_\Theta^-$ . Condition (3) is immediate from the definition of  $\mathfrak{l}_x$ .

Now we can state the conjugation result.

**Theorem 3.2** *The map  $\psi : V_\Theta(H, w) \rightarrow \mathbb{F}_\Theta$  is  $L_H$ -equivariant local diffeomorphism and satisfies:*

- i) *Its restriction to a neighborhood of the zero section is a diffeomorphism onto a neighborhood of  $\text{fix}_\Theta(H, w)$ .*
- ii) *Its restriction to  $V_\Theta^-(H, w)$  and  $V_\Theta^+(H, w)$  are diffeomorphisms onto  $\text{st}_\Theta(H, w)$  and  $\text{un}_\Theta(H, w)$ , respectively.*

**Proof:** The  $L_H$ -equivariance follows by construction and it is also immediate that  $\psi$  is the identity on the null section. Next we show that  $\psi$  is a local diffeomorphism. By the  $L_H$ -equivariance and the inverse function theorem, it is enough to show that the differential  $d_v\psi$  is an isomorphism for  $v = \tilde{X}(wb_\Theta)$ , where  $X \in \mathfrak{l}_{wb_\Theta}$ . We first note that the dimensions of the manifolds  $V_\Theta(H, w)$  and  $\mathbb{F}_\Theta$  are the same, since  $V_\Theta(H, w)$  is a normal bundle. Hence we only need to show that the differential is surjective. Taking  $Z \in \mathfrak{l}_{wb_\Theta}^0 = w\mathfrak{n}_\Theta^- \cap \mathfrak{z}_H$ , we have that

$$u(t) = \tilde{X}(\exp(tZ)wb_\Theta)$$

is a curve that is transversal to the fibers of  $V_\Theta(H, w)$ . Thus we can compute

$$\left. \frac{d}{dt} \right|_{t=0} \psi(u(t)) = \left. \frac{d}{dt} \right|_{t=0} \exp(X) \exp(tZ)wb_\Theta = d_{wb_\Theta} \exp(X) \tilde{Z}(wb_\Theta).$$

On the other hand, taking  $Y \in \mathfrak{l}_{wb_\Theta}$ , we have that

$$v(t) = (\log(\exp(X) \exp(tY)))^\sim(wb_\Theta)$$

is a curve of  $V_\Theta(H, w)$  in the fiber over  $wb_\Theta$ . Thus we can compute

$$\left. \frac{d}{dt} \right|_{t=0} \psi(v(t)) = \left. \frac{d}{dt} \right|_{t=0} \exp(X) \exp(tY)wb_\Theta = d_{wb_\Theta} \exp(X) \tilde{Y}(wb_\Theta).$$

Since  $d_{wb_\Theta} \exp(X)$  is an isomorphism, the dimension of the image of  $d_v\psi$ , where  $v = \tilde{X}(wb_\Theta)$ , is greater than the dimension of the space spanned at  $wb_\Theta$  by the induced vectors fields of  $\mathfrak{l}_{wb_\Theta} \oplus \mathfrak{l}_{wb_\Theta}^0$ . Since this is precisely the dimension of  $\mathbb{F}_\Theta$  it follows that  $d_v\psi$  is surjective.

For item (i), suppose by contradiction that there is no neighborhood of  $\text{fix}_\Theta(H, w)$  in  $V_\Theta(H, w)$  in which  $\psi$  is injective. Then we have sequences  $u_k, v_k \in V_\Theta(H, w)$  such that  $u_k \neq v_k$ ,  $\psi(u_k) = \psi(v_k)$  and  $u_k, v_k \rightarrow \text{fix}_\Theta(H, w)$  when  $k \rightarrow \infty$ . Since  $\text{fix}_\Theta(H, w)$  is compact, we can take a compact neighborhood of the null section in  $V_\Theta(H, w)$ . Thus we have, taking subsequences, that  $u_k \rightarrow x$ ,  $v_k \rightarrow y$ , where  $x, y \in \text{fix}_\Theta(H, w)$ . But then

$$x = \psi(x) = \lim_k \psi(u_k) = \lim_k \psi(v_k) = \psi(y) = y.$$

Since  $u_k \neq v_k$  the map  $\psi$  fails to be locally injective around  $x = y$ , contradicting the local diffeomorphism property already proved.

For item (ii) we first compute the image of  $\psi$  restricted to  $V_{\Theta}^{-}(H, w)$ . Note that

$$V_{\Theta}^{-}(H, w)_{wb_{\Theta}} = \mathfrak{n}_H^{-} \cdot wb_{\Theta}$$

and thus

$$V_{\Theta}^{-}(H, w) = L_H \cdot V_{\Theta}^{-}(H, w)_{wb_{\Theta}} = L_H \cdot (\mathfrak{n}_H^{-} \cdot wb_{\Theta}).$$

Therefore, by the  $L_H$ -equivariance, we have that

$$\psi(V_{\Theta}^{-}(H, w)) = L_H N_H^{-} \cdot wb_{\Theta} = N_H^{-} \cdot \text{fix}_{\Theta}(H, w) = \text{st}_{\Theta}(H, w),$$

so that the map is onto. For the injectivity suppose that

$$g \exp(X) \cdot wb_{\Theta} = k \exp(Y) \cdot wb_{\Theta}$$

where  $g, k \in L_H$  and  $X, Y \in w\mathfrak{n}_{\Theta}^{-} \cap \mathfrak{n}_H^{-}$ . Applying  $\exp tH$  to both sides and making  $t \rightarrow +\infty$ , the left hand side converges to  $g \cdot wb_{\Theta}$  and the right hand side to  $k \cdot wb_{\Theta}$ . Hence,  $g \cdot wb_{\Theta} = k \cdot wb_{\Theta}$ . Hence  $g^{-1}k \in wP_{\Theta}w^{-1}$ . It follows that

$$\exp(-X) \exp(\text{Ad}(g^{-1}k)Y) \in w(P_{\Theta} \cap N_{\Theta}^{-})w^{-1} = 1.$$

Since  $\exp$  restricted to  $\mathfrak{n}^{-}$  is a diffeomorphism, it follows that  $\text{Ad}(g)X = \text{Ad}(h)Y$  and  $(\text{Ad}(g)X)^{\sim}(g \cdot wb_{\Theta}) = (\text{Ad}(h)Y)^{\sim}(k \cdot wb_{\Theta})$ . That is, the map is injective. The proof for  $V_{\Theta}^{+}(H, w)$  is analogous.  $\square$

**Remark:** The bundles  $V_{\Theta}^{\pm}(H, w)$  as well as the stable and unstable sets  $\text{st}_{\Theta}(H, w)$  and  $\text{un}_{\Theta}(H, w)$  are  $Z_H$ -invariant. Despite this, the conjugation  $\psi$  is not in general  $Z_H$ -equivariant because it may not be true that the subspace

$$\mathfrak{l}_{wb_{\Theta}} = w\mathfrak{n}_{\Theta}^{-} \cap (\mathfrak{n}_H^{-} \oplus \mathfrak{n}_H^{+})$$

is invariant under the isotropy at  $wb_{\Theta}$  of the  $Z_H$ -action. An example of the non invariance is given next in the maximal flag manifold ( $\Theta = \emptyset$ ).

**Example:** The isotropy at  $wb$  of the  $Z_H$ -action is  $Z_H \cap P^w$  whose Lie algebra is

$$\mathfrak{z}_H \cap w\mathfrak{p} = w \left( \sum_{\beta} \mathfrak{g}_{\beta} \right)$$

with the sum extended to the roots  $\beta \geq 0$  such that  $\beta(w^{-1}H) = 0$ . On the other hand

$$\mathfrak{l}_{wb} = w\mathfrak{n}^- \cap (\mathfrak{n}_H^- \oplus \mathfrak{n}_H^+) = w(\sum_{\alpha} \mathfrak{g}_{\alpha})$$

with the sum over  $\alpha < 0$  with  $\alpha(w^{-1}H) \neq 0$ . Clearly  $\mathfrak{l}_{wb}$  is normalized by  $\mathfrak{z}_H \cap w\mathfrak{p}$  if it is invariant by  $Z_H \cap P^w$ .

Now let  $X_{\alpha} \in \mathfrak{g}_{\alpha}$ , where  $\alpha < 0$ ,  $\alpha(w^{-1}H) \neq 0$ , and  $X_{\beta} \in \mathfrak{g}_{\beta}$ , where  $\beta \geq 0$ ,  $\beta(w^{-1}H) = 0$ , be such that  $[X_{\alpha}, X_{\beta}] \neq 0$ . If  $\mathfrak{z}_H \cap w\mathfrak{p}$  normalizes  $\mathfrak{l}_{wb}$  then

$$w[X_{\alpha}, X_{\beta}] = [wX_{\alpha}, wX_{\beta}] \in \mathfrak{l}_{wb}$$

which happens if, and only if,

$$(\alpha + \beta)(w^{-1}H) \neq 0 \quad \text{and} \quad \alpha + \beta < 0,$$

since  $[X_{\alpha}, X_{\beta}] \in \mathfrak{g}_{\alpha+\beta}$ . Keeping this in mind, our example is given for  $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$  with the usual choices of chambers and roots. Let  $w \in \mathcal{W}$  be the permutation  $w = (23) = w^{-1}$ , and

$$H = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix}, \quad X_{\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_{\beta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where  $\alpha = \alpha_{21} = -\alpha_{12}$ ,  $\beta = \alpha_{13} = \alpha_{12} + \alpha_{23}$ . Thus  $\alpha + \beta = \alpha_{23} > 0$ ,  $\alpha < 0$  with  $\alpha(w^{-1}H) = 3 \neq 0$ ,  $\beta > 0$  with  $\beta(w^{-1}H) = 0$  and  $[X_{\alpha}, X_{\beta}] \neq 0$ . By the general remarks above it follows that  $\mathfrak{z}_H \cap w\mathfrak{p}$  does not normalize  $\mathfrak{l}_{wb}$  for this choice of  $H$  and  $w$ .

In conclusion we mention some cases where the conjugation is  $Z_H$  invariant.

**Corollary 3.3** *If  $H$  is regular then  $Z_H = L_H$  so that  $\psi$  is  $Z_H$ -equivariant.*

Another case where  $Z_H$ -invariance holds is for the attractor fixed point component  $\text{fix}_{\Theta}(H, 1)$  when the flag manifold  $\mathbb{F}_{\Theta}$  either projects onto  $\mathbb{F}_{\Theta(H)}$  or  $\mathbb{F}_{\Theta(H)}$  projects onto  $\mathbb{F}_{\Theta}$ .

**Corollary 3.4** *Let  $w = 1$  and suppose that*

$$\Theta \subset \Theta(H) \quad \text{or} \quad \Theta(H) \subset \Theta.$$

*Then  $\psi : V_{\Theta}(H, 1) = V_{\Theta}(H, 1)^- \rightarrow \mathbb{F}_{\Theta}$  is  $Z_H$ -equivariant and is a global diffeomorphism onto the open and dense subset  $\text{st}_{\Theta}(H, 1)$ .*

**Proof:** To prove  $Z_H$ -invariance it is enough to check that the subalgebra

$$\mathfrak{n}_\Theta^- \cap \mathfrak{n}_H^-$$

is invariant under  $\text{Ad}(Z_H \cap P_\Theta)$ . In general  $\mathfrak{n}_H^-$  is  $\text{Ad}(Z_H)$ -invariant. Now, if  $\Theta(H) \subset \Theta$  then  $Z_H \subset Z_\Theta$ , hence  $Z_H \cap P_\Theta \subset Z_\Theta$  which normalizes  $\mathfrak{n}_\Theta^-$ , so that  $\mathfrak{n}_\Theta^- \cap \mathfrak{n}_H^-$  is  $\text{Ad}(Z_H \cap P_\Theta)$ -invariant. On the other hand when  $\Theta \subset \Theta(H)$  then  $\mathfrak{n}_H^- \subset \mathfrak{n}_\Theta^-$  so that  $\mathfrak{n}_\Theta^- \cap \mathfrak{n}_H^- = \mathfrak{n}_H^-$ , which is normalized by  $Z_H$  showing the result in this case as well. The proof then follows closely the proof of Theorem 3.2 taking  $w = 1$  and replacing  $L_H$  by  $Z_H$ .  $\square$

**Remark:** A linearization similar to the above one was already given in Proposition 3.6 of [9]. However that linearization is not suitable for our purposes for two reasons. First it does not cover (in the nonregular case) the whole fixed point component, but only a dense subset of it. Also, it is  $wZ_Hw^{-1}$ -equivariant instead of  $Z_H$ -equivariant as we will need latter.

### 3.3 Fixed point components as flag manifolds

It is an interesting fact that each fixed point component  $\text{fix}_\Theta(H, w)$  in  $\mathbb{F}_\Theta$  is actually diffeomorphic to a flag manifold of a certain subgroup of  $G$ . In fact, for  $\Delta \subset \Sigma$  let  $\mathfrak{g}(\Delta)$  be the semi-simple subalgebra of  $\mathfrak{g}$  of type  $\Delta$ . For  $H_0 \in \mathfrak{a}(\Delta)$  let  $\mathbb{F}(\Delta)_{H_0}$  be the flag manifold of  $\mathfrak{g}(\Delta)$  of type  $H_0$ . Note that we do not need to worry in which chamber of  $\mathfrak{a}(\Delta)$  the element  $H_0$  lies. Denote by  $\pi_\Delta : \mathfrak{a} \rightarrow \mathfrak{a}(\Delta)$  the orthogonal projection.

**Proposition 3.5** *Let  $H \in \text{cl}\mathfrak{a}^+$ ,  $w \in W$ . Put  $\Delta = \Theta(\phi)$  and take  $H_\Theta \in \text{cl}\mathfrak{a}^+$  such that  $\Theta(H_\Theta) = \Theta$ . Then we have a diffeomorphism*

$$\text{fix}(H, w)_\Theta \simeq \mathbb{F}(\Delta)_{H_0},$$

where  $H_0$  is the orthogonal projection of  $wH_\Theta$  in  $\mathfrak{a}(\Delta)$ . Furthermore, in the maximal flag manifold  $\mathbb{F}$ , there is a  $K_H$ -equivariant diffeomorphism between any two  $\text{fix}(H, w)$ ,  $w \in W$ , which are thus diffeomorphic to the maximal flag manifold of  $\mathfrak{g}(\Delta)$ .

**Proof:** Decompose  $wH_\Theta = H_0 + H_1$ , where  $H_0 \in \mathfrak{a}(\Delta)$ ,  $H_1 \in \mathfrak{a}_\Delta$ . Using that  $K_H = K(\Delta)M$  we have that

$$\text{fix}_\Theta(H, w) = K_H w b_\Theta = K(\Delta) w b_\Theta.$$



Using that  $K_\Theta = K_{H_\Theta}$  it follows that the isotropy of  $K$  at  $wb_\Theta$  is  $K \cap wP_\Theta w^{-1} = wK_\Theta w^{-1} = K_{wH_\Theta}$ . So the isotropy of  $K(\Delta)$  at  $wb_\Theta$  is  $K(\Delta)_{wb_\Theta} = K(\Delta)_{H_0}$ , where the last equality follows from  $K(\Delta) \subset K_\Delta$  so it centralizes  $H_1 \in \mathfrak{a}_\Delta$ . It follows that

$$\text{fix}_\Theta(H, w) \rightarrow \mathfrak{s}(\Delta), \quad kwb_\Theta \mapsto \text{Ad}(k)H_0, \quad \text{where } k \in K(\Delta),$$

is an embedding. Equation (1) applied to  $\mathfrak{g}(\Delta)$  gives that the image of this embedding is diffeomorphic to the flag manifold  $\mathbb{F}(\Delta)_{H_0}$ . This proves the first assertion. For the second assertion, we note that

$$\mathbb{F} \rightarrow \mathbb{F}, \quad kb \mapsto kwb, \quad \text{where } k \in K,$$

is a well defined  $K$ -invariant diffeomorphism of  $\mathbb{F}$ . In fact, if  $kb = k'b$  then  $k' = km$ ,  $m \in M$ . Since  $w$  normalizes  $M$ , it follows that  $k'wb = kwb$ . The above map clearly restricts to a  $K_H$ -invariant diffeomorphism between  $\text{fix}(H, 1) = K_H b$  and  $\text{fix}(H, w) = K_H wb$ . By the first part of the proof, we have that  $\text{fix}(H, 1) \simeq \mathbb{F}(\Delta)_{\pi_\Delta(H_\Theta)}$ , where  $H_\Theta$  is regular since we are in the maximal flag manifold. We claim that  $\pi_\Delta(H_\Theta) = H_0$  is regular in  $\mathfrak{a}(\Delta)$ , which implies that  $\mathbb{F}(\Delta)_{H_0}$  is the maximal flag manifold of  $\mathfrak{g}(\Delta)$ . In fact, for all  $\alpha \in \langle \Delta \rangle$  we have  $\alpha|_{\mathfrak{a}_\Delta} = 0$  so that  $\alpha(H_0) = \alpha(H_\Theta) \neq 0$ .  $\square$

### 3.4 Schubert cells

We recall some facts about the closure of the stable sets  $\text{st}_\Theta(H, w)$ . If  $H$  is regular then the closure  $\text{cl}(\text{st}_\Theta(H, w))$  is known as a Schubert cell in the flag manifold  $\mathbb{F}_\Theta$ . In this case a result that goes back to Borel-Tits [4] ensures that a Schubert cell is a union of Bruhat cells. Precisely,

$$\text{cl}(\text{st}_\Theta(H, w)) = \bigcup_{s \geq w} \text{st}_\Theta(H, s) = \bigcup_{s \geq w} N^-(s \cdot b_\Theta) \quad (4)$$

where the order in  $\mathcal{W}$  is Bruhat-Chevalley. For nonregular  $H$  the same equality is true because a fixed point component  $\text{fix}_\Theta(H, w)$  is a  $K_H$ -orbit and

$$\text{st}_\Theta(H, w) = N_H^- K_H w b_\Theta = K_H N^- w b_\Theta,$$

that is,  $\text{st}_\Theta(H, w)$  is the union of  $K_H$ -translations of Bruhat cells. Hence (4) holds as well after taking closures, since  $K_H$  is compact.

These descriptions of Schubert cells are group theoretic. On the other hand on the maximal flag manifold there is the following alternative description of [21]: Fix a simple system of roots  $\Sigma$ , and for a finite sequence  $\alpha_1, \dots, \alpha_n$  in  $\Sigma$  we let  $s_1, \dots, s_n$  be the reflections with respect to these roots. Then we write  $\mathbb{F}_i = \mathbb{F}_{\{\alpha_i\}}$  and put  $\pi_i : \mathbb{F} \rightarrow \mathbb{F}_i$  for the canonical projection. Accordingly, we write  $\gamma_i = \pi_i^{-1}\pi_i$  for the map that to a subset  $A \subset \mathbb{F}$  associates the union of the fibers crossing  $A$ . Then it follows by the results of [21] that

$$\text{cl}(\text{st}(H, w)) = \gamma_1 \cdots \gamma_n (\text{fix}(H, w^-)),$$

where  $\gamma_1, \dots, \gamma_n$  is taken from a reduced expression  $w^- w = s_n \cdots s_1$ .

## 4 Morse decomposition on flag bundles

From now on we fix the setup stated in the introduction: A principal bundle  $\pi : Q \rightarrow X$  with semi-simple structural group  $G$  and compact Hausdorff base space  $X$ . A right invariant flow  $\phi_t$  on  $Q$  which is chain transitive on  $X$ . The flow  $\phi_t$  induces a flow in the flag bundle  $\mathbb{E}_\Theta$ . The following result, proved in [5] and [17], gives the chain components and Morse decomposition of this induced flow.

**Theorem 4.1** *The flow  $\phi_t$  on the flag bundle  $\mathbb{E}_\Theta \rightarrow X$  admits a finest Morse decomposition whose Morse components are determined as follows.*

- i) *There exists  $H_\phi \in \text{cl}\mathfrak{a}^+$  and a  $\phi_t$ -invariant  $Z_{H_\phi}$ -reduction  $Q_\phi$  of  $Q$  such that the Morse components are parameterized by the Weyl group  $\mathcal{W}$  and each component  $\mathcal{M}_\Theta^w$  is given by*

$$\mathcal{M}_\Theta^w = Q_\phi \cdot \text{fix}(H_\phi, w)_\Theta = Q_\phi \cdot w b_\Theta.$$

- ii) *There is just one attractor component  $\mathcal{M}_\Theta^+ = \mathcal{M}_\Theta^1$  and only one repeller  $\mathcal{M}_\Theta^- = \mathcal{M}_\Theta^{w^-}$ .*

- iii) *The dynamical ordering of the Morse components is given by the algebraic Bruhat-Chevalley order of  $\mathcal{W}$ .*

- iv) *There exists a  $K_{H_\phi}$ -reduction  $R_\phi$  of  $Q_\phi$  (which is not necessarily  $\phi_t$ -invariant) such that each Morse component is given by the orbit*

$$\mathcal{M}_\Theta^w = R_\phi \cdot \text{fix}(H_\phi, w)_\Theta = R_\phi \cdot w b_\Theta.$$

We say that  $H_\phi$  a characteristic element for the flow. The set of simple roots  $\Theta(\phi) = \Theta(H)$  is called the parabolic type of the flow and the  $\phi_t$ -invariant  $Z_H$ -reduction  $Q_\phi$  is called a block reduction of the flow (cf. [22]). When the characteristic element  $H_\phi$  is regular we say that the flow  $\phi_t$  is regular as well.

In the sequel we say that a subset  $A \subset \mathbb{E}_\Theta$  is a section over  $X$  if it is the image of a continuous section of the bundle  $\mathbb{E}_\Theta \rightarrow X$ . If a subset is a section then it meets each fiber in a singleton. For example the  $\mathcal{M}_{\Theta(\phi)}^+$  is a section of the flag bundle  $\mathbb{E}_{\Theta(\phi)} \rightarrow X$  associated to the parabolic type of  $\phi$ .

Next we show the interesting fact that the Morse components  $\mathcal{M}_\Theta^w$  can be viewed as a flag subbundles of  $\mathbb{E}_\Theta$  (cf. Proposition 3.5 above).

**Proposition 4.2** *Put  $\Delta = \Theta(\phi)$  and take  $H_\Theta \in \text{cla}^+$  such that  $\Theta(H_\Theta) = \Theta$ . Then the following statements hold.*

- i) *A Morse component  $\mathcal{M}_\Theta^w$  is an associated bundle of  $Q_\phi \rightarrow X$  with typical fiber  $\mathbb{F}(\Delta)_{H_0}$  where  $H_0$  is the orthogonal projection of  $wH_\Theta$  in  $\mathfrak{a}(\Delta)$ .*
- ii) *In the maximal flag bundle  $\mathbb{E}$ , there is a bundle homeomorphism between any two Morse components  $\mathcal{M}^w$ ,  $w \in \mathcal{W}$ . The typical fiber of any one of these bundles is the maximal flag manifold  $\mathbb{F}(\Delta)$  of  $G(\Delta)$ .*
- iii) *If the flow is regular, then each Morse component  $\mathcal{M}_\Theta^w$  is a section over  $X$ .*

**Proof:** By Theorem 4.1, we have  $\mathcal{M}_\Theta^w = Q_\phi \cdot wb_\Theta$  hence  $\mathcal{M}^w \rightarrow X$  is a fiber bundle associated to  $Q_\phi \rightarrow X$  with typical fiber  $Z_H wb_\Theta = \text{fix}_\Theta(H, w)$ . Using that  $\Delta = \Theta(\phi) = \Theta(H_\phi)$  the result follows from the first part of Proposition 3.5. In order to construct a fiber-bundle homeomorphism between the components  $\mathcal{M}^w \subset \mathbb{E}$  we consider the  $K_H$  reduction  $R_\phi$  of  $Q_\phi$  given by item (iv) of Theorem 4.1 and also the  $K_H$ -equivariant diffeomorphism between any two  $\text{fix}(H, w)$  given by Proposition 3.5. Plugging this diffeomorphism on the fibers of the flag bundles through the  $R_\phi$  reduction yields the desired bundle homeomorphism. Finally in the regular case the typical fibers are just a points.  $\square$

## 5 Stable and unstable sets

The stable and unstable sets

$$\text{st}(\mathcal{M}) = \{x \in E : \omega(x) \subset \mathcal{M}\} \quad \text{un}(\mathcal{M}) = \{x \in E : \omega^*(x) \subset \mathcal{M}\}$$

of a Morse component  $\mathcal{M}$  in a flag bundle are given in a similar fashion as the components themselves. Namely, by plugging fiberwise stable and unstable sets on the flag manifolds. We work out the stable sets. The unstable ones are obtained by symmetry.

We start by recalling a result of [5]. Following the notation of Subsection 3.4 let  $s_1, \dots, s_n$  be reflections with respect to the simple roots  $\alpha_1, \dots, \alpha_n$  and write  $\mathbb{E}_i = \mathbb{E}_{\{\alpha_i\}}$  for the corresponding flag bundles. Put  $\pi_i : \mathbb{E} \rightarrow \mathbb{E}_i$  for the canonical projection and  $\gamma_i = \pi_i^{-1}\pi_i$  for the map that exhausts fibers. Note that each map  $\gamma_i$  preserves the fibers of  $\mathbb{E} \rightarrow X$ .

The following statement is Proposition 9.9 of [5]. It characterizes the domains of attraction in the maximal flag bundle.

**Proposition 5.1** *The domain of attraction of  $\mathcal{M}^w$  is given by*

$$\mathcal{A}(\mathcal{M}^w) = \gamma_1 \cdots \gamma_n(\mathcal{M}^-), \quad (5)$$

where  $\gamma_1, \dots, \gamma_n$  is taken from any reduced expression  $w^-w = s_n \cdots s_1$ .

Now define

$$\mathbb{S}_\Theta^w = Q_\phi \cdot \text{st}_\Theta(H, w) = \{q \cdot b \in \mathbb{E}_\Theta : q \in Q_\phi, b \in \text{st}_\Theta(H, w)\}. \quad (6)$$

We plan to prove that  $\mathbb{S}_\Theta^w$  is the stable set of the Morse component  $\mathcal{M}_\Theta^w$ . Note that  $\mathbb{S}_\Theta^w$  is  $\phi_t$ -invariant because  $Q_\phi$  is  $\phi_t$ -invariant and  $\text{st}_\Theta(H, w)$  is  $Z_H$ -invariant. Also,  $\mathbb{S}_\Theta^w$  contains  $\mathcal{M}_\Theta^w$ . The next result collects some properties of these sets.

**Proposition 5.2** *Fix  $\Theta \subset \Sigma$  and  $w \in \mathcal{W}$ . Then the following statements hold.*

- i)  $(\mathbb{S}_\Theta^w)_{\pi(q)} = q \cdot \text{st}_\Theta(H, w)$  for a fixed  $q \in Q_\phi$ .
- ii) We have that  $\text{cl}(\mathbb{S}_\Theta^w) = \bigcup_{s \geq w} \mathbb{S}_\Theta^s$ . Furthermore, if  $\mathcal{M}_\Theta^s \subset \text{cl}(\mathbb{S}_\Theta^w)$  then  $\mathcal{M}_\Theta^s = \mathcal{M}_\Theta^{\bar{w}}$ , where  $\bar{w} \geq w$ .

iii) The flag bundle decomposes as  $\mathbb{F}_\Theta = \coprod \{\mathbb{S}_\Theta^w : w \in \mathcal{W}_H \setminus \mathcal{W}/\mathcal{W}_\Theta\}$ .

iv) In the maximal flag bundle  $\text{cl}(\mathbb{S}^w) = \mathcal{A}(\mathcal{M}^w) = \gamma_1 \cdots \gamma_n(\mathcal{M}^-)$ .

v) Each  $\mathbb{S}_\Theta^w$  is homeomorphic to a vector bundle over  $\mathcal{M}_\Theta^w$ .

**Proof:** Fixing  $q \in Q_\phi$  we have that  $(Q_\phi)_{\pi(q)} = qZ_H$  and, since  $\text{st}_\Theta(H, w)$  is  $Z_H$ -invariant we obtain

$$(\mathbb{S}_\Theta^w)_{\pi(q)} = (Q_\phi)_{\pi(q)} \cdot \text{st}_\Theta(H, w) = q \cdot \text{st}_\Theta(H, w),$$

which proves (i). For (ii) we take  $q \in Q_\phi$  and prove first that

$$\text{cl}(\mathbb{S}_\Theta^w)_{\pi(q)} = q \cdot \text{cl}(\text{st}_\Theta(H, w)).$$

The inclusion “ $\supset$ ” is immediate. For the other inclusion take  $\xi \in \text{cl}(\mathbb{S}_\Theta^w)$  such that  $\pi(\xi) = \pi(q) = x \in X$ . Then there is a net  $\xi_k \in \mathbb{S}_\Theta^w$  such that  $\xi_k \rightarrow \xi$  so that  $x_k := \pi(\xi_k) \rightarrow \pi(\xi) = x$ . Let  $\chi : U \rightarrow Q_\phi$  be a continuous local section of  $Q_\phi \subset Q$  defined in a neighborhood  $U$  of  $x$  and take  $k$  sufficiently large so that  $x_k \in U$ . From item (i) it follows that  $\xi_k = \chi(x_k) \cdot v_k$  where  $v_k \in \text{st}_\Theta(H, w)$ , so that

$$v_k = \chi(x_k)^{-1} \xi_k \rightarrow \chi(x)^{-1} \xi \in \text{cl}(\text{st}_\Theta(H, w)),$$

which proves that  $\xi \in q \cdot \text{cl}(\text{st}_\Theta(H, w))$  and establishes the other inclusion. Using the first statement and equation (4) for nonregular  $H$ , it follows that

$$\text{cl}(\mathbb{S}_\Theta^w)_{\pi(q)} = q \cdot \bigcup_{\overline{w} \geq w} \text{st}_\Theta(H, \overline{w}) = \bigcup_{\overline{w} \geq w} q \cdot \text{st}_\Theta(H, \overline{w}) = \bigcup_{\overline{w} \geq w} (\mathbb{S}_\Theta^{\overline{w}})_{\pi(q)}.$$

Since  $q \in Q_\phi$  is arbitrary, this proves the first statement of (ii). The second statement of (ii) and (iii) follow by (i) and the Bruhat decomposition of the typical fiber  $\mathbb{F}_\Theta$ .

Statement (iv) follows from Proposition 5.1 and the characterizations of the Schubert cells in Subsection 3.4.

In order to prove (v), we first take the  $K_H$ -reduction  $R_\phi$  of  $Q_\phi$  provided by Theorem 4.1 (iv) and the  $L_H$ -invariant vector bundle  $\pi_\Theta : V_\Theta^-(H, w) \rightarrow \text{fix}_\Theta(H, w)$  of Section 3. Since  $K_H \subset L_H$  we can define the associated bundle of  $R_\phi$  given by

$$\mathcal{V} = R_\phi \times_{K_H} V_\Theta^-(H, w) \rightarrow X.$$

Defining the projection  $\pi_\Theta : \mathcal{V} \rightarrow \mathbb{E}_\Theta$ ,  $q \cdot v \mapsto q \cdot \pi_\Theta(v)$  we have, by of Theorem 4.1 (iv), that  $\pi_\Theta(\mathcal{V}) = \mathcal{M}_\Theta^w$ . This shows that  $\mathcal{V}$  can be viewed as a vector bundle over  $\mathcal{M}_\Theta^w$ . Recall the  $K_H$ -equivariant diffeomorphism  $\psi : V_\Theta^-(H, w) \rightarrow \text{st}_\Theta(H, w)$  of Section 3 and consider the homeomorphism

$$\Psi : \mathcal{V} \rightarrow \mathbb{E}_\Theta, \quad q \cdot v \mapsto q \cdot \psi(v),$$

whose image is precisely  $R_\phi \cdot \text{st}_\Theta(H, w)$ . Using that  $Q_\phi = R_\phi Z_H$  and that  $\text{st}_\Theta(H, w)$  is  $Z_H$ -invariant it follows that  $\Psi$  has image  $\mathbb{S}_\Theta^w$ .  $\square$

**Theorem 5.3** *The stable set of the Morse component  $\mathcal{M}_\Theta^w \subset \mathbb{E}_\Theta$  is*

$$\text{st}(\mathcal{M}_\Theta^w) = \mathbb{S}_\Theta^w,$$

*for every  $w \in \mathcal{W}$ ,  $\Theta \in \Sigma$ . In particular, each  $\text{st}(\mathcal{M}_\Theta^w)$  is homeomorphic to a vector bundle over  $\mathcal{M}_\Theta^w$ .*

**Proof:** We work first in the maximal flag bundle  $\mathbb{E}$  where we omit the subscript  $\Theta$ . In this case the proof is by induction on the length of  $w$ .

First we observe that if  $w$  has maximal length, that is,  $w = w^-$  then  $\mathcal{M}^w$  is the repeller component by Theorem 4.1. So that  $\text{st}(\mathcal{M}^w) = \mathcal{M}^w = \mathbb{S}^w$ , since  $\text{st}_\Theta(H, w^-) = w^-b$ . Now suppose that the result is true for all  $s \in \mathcal{W}$  with length strictly greater than the length of  $w$ . Then  $\text{st}(\mathcal{M}^s) = \mathbb{S}^s$  for all  $s > w$  by the inductive hypothesis.

By (iv) and (ii) of the above proposition the domain of attraction  $\mathcal{A}(\mathcal{M}^w)$  of  $\mathcal{M}^w$  is the invariant set

$$\bigcup_{s \geq w} \mathbb{S}^s.$$

By the definitions, the stable set of  $\mathcal{M}^w$  is contained in  $\mathcal{A}(\mathcal{M}^w)$ . It follows that if we take

$$x \in \mathcal{A}(\mathcal{M}^w) \setminus \mathbb{S}^w = \bigcup_{s > w} \mathbb{S}^s,$$

then its  $\omega$ -limit set is contained in  $\mathcal{M}^s \neq \mathcal{M}^w$  with  $s > w$ . Hence the stable set of  $\mathcal{M}^w$  is contained in  $\mathbb{S}^w$ . Since  $\mathcal{A}(\mathcal{M}^w)$  is closed and invariant, it contains the  $\omega$ -limit sets of its points, so that the stable set of  $\mathcal{M}^w$  is indeed  $\mathbb{S}^w$ .

Now let  $\Theta \subset \Sigma$  be arbitrary and let  $\pi_\Theta : \mathbb{E} \rightarrow \mathbb{E}_\Theta$  be the natural projection between these flag subbundles. Then

$$\mathbb{S}_\Theta^w = \pi_\Theta(\mathbb{S}^w) = \pi_\Theta(\text{st}(\mathcal{M}^w)) \subset \text{st}(\mathcal{M}_\Theta^w),$$

where the last inclusion follows from the equivariance of the flow  $\phi_t$  with respect to the projection  $\pi_\Theta$ . Since both  $\mathbb{S}_\Theta^w$  and  $\text{st}(\mathcal{M}_\Theta^w)$  partition  $\mathbb{E}_\Theta$  into disjoint sets this inclusion implies the equality  $\mathbb{S}_\Theta^w = \text{st}(\mathcal{M}_\Theta^w)$ , which proves the result.  $\square$

**Remark:** Note that, in the present situation, the stable set of  $\mathcal{M}^w$  is open and dense in its domain of attraction  $\mathcal{A}(\mathcal{M}^w)$ . These results on stable bundles extend to flows on flag bundles the Bruhat decomposition for the action of semi-simple elements in flag manifolds (see e.g. [9]).

Defining  $\mathbb{U}_\Theta^w = Q_\phi \cdot \text{un}_\Theta(H, w)$ , it follows by entirely analogous arguments that

$$\text{un}(\mathcal{M}_\Theta^w) = \mathbb{U}_\Theta^w.$$

## 6 Linearizations around Morse components

Now we write down linearizations of the flow around the Morse components on the flag bundles. Take a characteristic element  $H_\phi$  for the flow and recall the subgroup  $L_{H_\phi} = K_{H_\phi} A_{H_\phi} \subset Z_{H_\phi}$  of Section 3.

**Definition 6.1** *We say that the flow  $\phi_t$  admits a conformal reduction if there exists a subbundle  $C_\phi \subset Q_\phi$  with structural group  $L_H$  which is  $\phi_t$ -invariant.*

In general a flow does not admit a conformal reduction contrary to the  $Z_H$ -block reduction  $Q_\phi$  that holds by Theorem 4.1. To construct linearizations on the flag bundles we make the additional assumption that conformal reduction exists. The reason for this assumption is that linearization on flag manifolds built in Subsection 3.2 is  $L_H$ -equivariant but not in general  $Z_H$ -equivariant. We note however that if the flow is regular, then  $L_{H_\phi} = Z_{H_\phi} = MA$  and the conformal reduction is granted by the block reduction. This regularity of the flow happens, for example, when the flow has simple Lyapunov spectrum for every ergodic measure (as follows by the results of [22]).

When there exists a conformal reduction  $C_\phi$  it can replace the block reduction  $Q_\phi$  in Theorem 4.1. For instance, the Morse components are given by

$$\mathcal{M}_\Theta^w = C_\phi \cdot \text{fix}_\Theta(H, w) = C_\phi \cdot wb_\Theta \quad (7)$$

which is contained in its stable bundle

$$\mathbb{S}_\Theta^w = C_\phi \cdot \text{st}_\Theta(H, w). \quad (8)$$

In fact,  $\mathcal{M}_\Theta^w = Q_\phi \cdot \text{fix}_\Theta(H, w)$  by Theorem 4.1 (i). But then it follows from the  $Z_H$ -invariance of  $\text{fix}_\Theta(H, w)$  and the equality  $Q_\phi = C_\phi \cdot Z_H$  that

$$\mathcal{M}_\Theta^w = C_\phi Z_H \cdot \text{fix}_\Theta(H, w) = C_\phi \cdot \text{fix}_\Theta(H, w).$$

Also,  $\text{fix}_\Theta(H, w) = L_H wb_\Theta$  so that  $\mathcal{M}_\Theta^w = C_\phi \cdot wb_\Theta$ . For the equality involving  $\mathbb{S}_\Theta^w$ , using the equation (6), we proceed in the same way as in the proof of the first equality above, since  $\text{st}_\Theta(H, w)$  is  $Z_H$  invariant.

The construction of the linearization on the flag bundles follows easily from the equivariance of the linearization on the flag manifolds. Assume there exists a conformal reduction  $C_\phi$  and fix  $w \in \mathcal{W}$ ,  $\Theta \subset \Sigma$ . Recall the  $L_H$ -invariant vector bundle  $\pi_\Theta : V_\Theta(H, w) \rightarrow \text{fix}_\Theta(H, w)$  of Section 3 and consider the associated bundle of  $C_\phi$  given by

$$\mathcal{V}_\Theta^w = C_\phi \times_{L_H} V_\Theta(H, w) \rightarrow X,$$

obtained by the left action of  $L_H$  on  $V_\Theta(H, w)$ . The bundle  $\mathcal{V}_\Theta^w$  is a vector bundle over  $\mathcal{M}_\Theta^w$  with projection

$$\pi_\Theta : \mathcal{V}_\Theta^w \rightarrow \mathbb{E}_\Theta, \quad q \cdot v \mapsto q \cdot \pi_\Theta(v).$$

(Note that by (7)  $q \cdot \pi_\Theta(v) \in \mathcal{M}_\Theta^w$  if  $q \in C_\phi$ .)

Now, recall the  $L_H$ -equivariant map  $\psi$  of Section 3 and consider the bundle map

$$\Psi : \mathcal{V}_\Theta^w \rightarrow \mathbb{E}_\Theta, \quad q \cdot v \mapsto q \cdot \psi(v). \quad (9)$$

It is well defined and its image contains  $\mathcal{M}_\Theta^w$ . Also,  $\phi_t$  induces a flow on associated bundle  $\mathcal{V}_\Theta^w$  by

$$\Phi_t(q \cdot v) = \phi_t(q) \cdot v.$$



Since the following diagram commutes

$$\begin{array}{ccc} \mathcal{V}_\Theta^w & \xrightarrow{\Phi_t} & \mathcal{V}_\Theta^w \\ \pi_\Theta \downarrow & & \downarrow \pi_\Theta \\ \mathcal{M}_\Theta^w & \xrightarrow{\phi_t} & \mathcal{M}_\Theta^w \end{array} \quad (10)$$

it follows that  $\Phi_t$  is a linear flow induced by  $\phi_t$  such that its induced flow on the base  $\mathcal{M}_\Theta^w$  is precisely the restriction of  $\phi_t$ , which is chain transitive. By of Theorem 3.2 (i) it follows that  $\Phi_t$  and  $\phi_t$  are conjugated by the equivariant map  $\Psi$  in a neighborhood of the null section.

We define the associated bundle of  $C_\phi$  given by

$$\mathcal{S}_\Theta^w = C_\phi \times_{L_H} V_\Theta^-(H, w), \quad (11)$$

which, in the same way as above, can be viewed as a vector bundle over  $\mathcal{M}_\Theta^w$ . By the equation (8), its image by  $\Psi$  is precisely the  $\phi_t$ -invariant set  $\mathbb{S}_\Theta^w$  which contains  $\mathcal{M}_\Theta^w$ . By item (ii) of the Theorem 3.2, it follows that  $\Phi_t$  and  $\phi_t$  are conjugated by the equivariant global homeomorphism  $\Psi$  restricted to the  $\phi_t$ -invariant vector bundle  $\mathcal{S}_\Theta^w$ , which from now on will be called the stable vector bundle of  $\mathcal{M}_\Theta^w$ . We collect the above results in the following proposition.

**Proposition 6.2** *Assume the existence of a conformal reduction  $C_\phi$ . Then the restriction of  $\phi_t$  to  $\mathbb{S}_\Theta^w$  is conjugate to a linear flow over the restriction of  $\phi_t$  to  $\mathcal{M}_\Theta^w$ .*

In order to define the unstable vector bundle we proceed analogously. Recalling the  $L_H$ -invariant subbundle  $V_\Theta^+(H, w)$  of Section 3, we define the  $\phi_t$ -invariant bundle of  $C_\phi$  given by

$$\mathcal{U}_\Theta^w = C_\phi \times_{L_H} V_\Theta^+(H, w).$$

Its image under  $\Psi$  is  $\text{un}(\mathcal{M}_\Theta^w) = \mathbb{U}_\Theta^w$ . We observe that from the definition of these vector bundles it follows from equation (2) the following  $\phi_t$ -invariant Whitney sum decomposition

$$\mathcal{V}_\Theta^w = \mathcal{S}_\Theta^w \oplus \mathcal{U}_\Theta^w \rightarrow \mathcal{M}_\Theta^w. \quad (12)$$

A case where the assumption of a conformal reduction is not needed is the following. If  $\Theta \subset \Theta(\phi)$  or  $\Theta(\phi) \subset \Theta$  then, by Corollary 3.4, the conjugation  $\psi$  around the attractor component is  $Z_H$ -equivariant. Hence, proceeding as above, we have the following.

**Corollary 6.3** *Assume that  $\Theta \subset \Theta(\phi)$  or  $\Theta(\phi) \subset \Theta$ . Then the restriction of  $\phi_t$  to  $\mathbb{S}_\Theta^+$  is conjugate to a linear flow over the restriction of  $\phi_t$  to the attractor  $\mathcal{M}_\Theta^+$ .*

## 7 Conley indices

In order to compute the Conley indices associated to the finest Morse decomposition of  $\mathbb{E}_\Theta$ , we assume, as in the previous section, the existence of a conformal reduction  $C_\phi$ . This condition will then be dropped for the computation of the Conley index of the attractor component.

**Proposition 7.1** *Assume the existence of a conformal reduction  $C_\phi$ . Then there exists a norm  $|\cdot|$  in the vector bundle  $\mathcal{V}_\Theta^w$  and a constant  $\alpha > 0$  such that*

- i)  $|\Phi_t(q \cdot v)| \leq e^{-\alpha t} |q \cdot v|$ , for  $q \cdot v \in \mathcal{S}_\Theta^w$ ,
- ii)  $|\Phi_t(q \cdot v)| \geq e^{\alpha t} |q \cdot v|$ , for  $q \cdot v \in \mathcal{U}_\Theta^w$ .

**Proof:** Let  $(\mathcal{S}_\Theta^w)_0$  be the null section of the stable vector bundle  $\mathcal{S}_\Theta^w \rightarrow \mathcal{M}_\Theta^w$  defined in the equation (11). Using the linearization  $\Psi$  given by Proposition 6.2 and using Theorem 5.3, it follows that, for each  $v \in \mathcal{S}_\Theta^w$ , we have  $\Phi_t(v) \rightarrow (\mathcal{S}_\Theta^w)_0$  when  $t \rightarrow \infty$ . Choosing a continuous vector bundle norm  $|\cdot|_1$  in  $\mathcal{S}_\Theta^w \rightarrow \mathcal{M}_\Theta^w$  we get from Fenichel's uniformity Lemma (see e.g. [2], Proposition 3.2) that there exists constants  $C, \alpha_1 > 0$  such that

$$|\Phi_t(q \cdot v)|_1 \leq C e^{-\alpha_1 t} |q \cdot v|_1,$$

for  $q \cdot v \in \mathcal{S}_\Theta^w$ . By Proposition 3.2 of [2] we can change the above norm to obtain  $C = 1$  in the above inequality, so we can assume without loss of generality that  $C = 1$ . In an entirely analogous fashion, arguing with the reversed time flow in the unstable vector bundle  $\mathcal{U}_\Theta^w \rightarrow \mathcal{M}_\Theta^w$ , we get a norm  $|\cdot|_2$  in  $\mathcal{U}_\Theta^w$  and a constant  $\alpha_2 > 0$  such that

$$|\Phi_t(q \cdot v)|_2 \geq e^{\alpha_2 t} |q \cdot v|_2,$$

for  $q \cdot v \in \mathcal{U}_\Theta^w$ . Using the Whitney sum decomposition  $\mathcal{V}_\Theta^w = \mathcal{S}_\Theta^w \oplus \mathcal{U}_\Theta^w \rightarrow \mathcal{M}_\Theta^w$  given in (12) it is clear that these two norms add up to a norm in  $|\cdot|$  in  $\mathcal{V}_\Theta^w$  which restricts to  $|\cdot|_1$  in  $\mathcal{S}_\Theta^w$  and to  $|\cdot|_2$  in  $\mathcal{U}_\Theta^w$ . Choosing  $\alpha := \min\{\alpha_1, \alpha_2\}$

we obtain the result.  $\square$

The previous result shows that the linearized flow in  $\mathcal{V}_\Theta^w$  is hyperbolic, so we can think of each Morse component  $\mathcal{M}_\Theta^w$  as a normally hyperbolic invariant set in  $\mathbb{E}_\Theta$ . For the attractor component, using Corollary 6.3 and proceeding as in the above proof, we can drop the assumption on the existence of a conformal reduction.

**Corollary 7.2** *Assume that  $\Theta \subset \Theta(\phi)$  or  $\Theta(\phi) \subset \Theta$ . Then there exists a norm  $|\cdot|$  in the vector bundle  $\mathcal{S}_\Theta^+$  and a constant  $\alpha > 0$  such that  $|\Phi_t(q \cdot v)| \leq e^{-\alpha t}|q \cdot v|$ , for  $q \cdot v \in \mathcal{S}_\Theta^+$ .*

Now we use the linearization  $\Psi : \mathcal{V}_\Theta^w \rightarrow \mathbb{E}_\Theta$  given in (9) and the norm in  $\mathcal{V}_\Theta^w$  given by Proposition 7.1 to construct an index pair for each  $\mathcal{M}_\Theta^w$ . Recall that, by item (i) of Theorem 3.2, it follows that  $\Phi_t$  and  $\phi_t$  are conjugated by the equivariant map  $\Psi$  in a neighborhood  $A$  of the null section. For  $q \in C_\phi$ ,  $v \in V_\Theta(H, w)$ , let  $q \cdot v^s$  and  $q \cdot v^u$  denote, respectively, the stable and the unstable components of the invariant Whitney sum decomposition of  $\mathcal{V}_\Theta^w$  given in (12). We can multiply the norm of  $\mathcal{V}_\Theta^w$  by a positive constant so that the set  $\{q \cdot v : |q \cdot v^s|, |q \cdot v^u| \leq 1\}$  is contained in  $A$ . We then define

$$\mathbb{B}_1 = \Psi(\{q \cdot v : |q \cdot v^s|, |q \cdot v^u| \leq 1\})$$

and

$$\mathbb{B}_0 = \Psi(\{q \cdot v : |q \cdot v^s| \leq 1, |q \cdot v^u| = 1\}).$$

**Proposition 7.3** *Assume the existence of a conformal reduction  $C_\phi$ . Then the pair  $(\mathbb{B}_1, \mathbb{B}_0)$  is an index pair for  $\mathcal{M}_\Theta^w$ .*

**Proof:** We need to verify the three properties of the definition of index pair (see Section 2.1). Property (1) is immediate, since clearly  $\text{cl}(\mathbb{B}_1 \setminus \mathbb{B}_0) = \mathbb{B}_1$  is a neighborhood of  $\mathcal{M}_\Theta^w$  which, from Proposition 7.1, contains no other invariant sets. In order to check property (2), we first note that

$$\phi_t(\Psi(q \cdot v)) = \Psi(\Phi_t(q \cdot v)).$$

Then, for  $\xi \in \mathbb{B}_0$  and  $t \geq 0$ , we have  $\phi_t(\xi) \in \mathbb{B}_1$  if and only if  $t = 0$ , since

$$|\Phi_t(q \cdot v^u)| \geq e^{\alpha t}|q \cdot v^u| = e^{\alpha t} \geq 1,$$

by Proposition 7.1. It remains to prove property (3). Thus let  $\xi \in \mathbb{B}_1$  be such that there exists  $t' > 0$  with  $\phi_{t'}(\xi) \notin \mathbb{B}_1$ . We define

$$t_m = \sup\{t : \phi_t(\xi) \in \mathbb{B}_1\}.$$

Since  $\mathbb{B}_1$  is closed, it follows that  $\phi_{t_m}(\xi) \in \mathbb{B}_1$ . We must show that  $\phi_{t_m}(\xi) \in \mathbb{B}_0$ . If we assume the contrary then  $\phi_{t_m}(\xi) = \Psi(q \cdot v)$  with  $|q \cdot v^s| \leq 1$  and  $|q \cdot v^u| < 1$ . For  $t > 0$  we have

$$\phi_{t_m+t}(\xi) = \phi_t(\Psi(q \cdot v)) = \Psi(\Phi_t(q \cdot v)).$$

But

$$|\Phi_t(q \cdot v^s)| \leq e^{-\alpha t} |q \cdot v^s| < 1.$$

Hence by continuity we also have  $|\Phi_t(q \cdot v^u)| < 1$  if  $t > 0$  small enough. This implies that  $\phi_{t_m+t}(\xi) \in \mathbb{B}_1$ , contradicting the definition of  $t_m$ .  $\square$

Now we relate the Conley index of  $\mathcal{M}_\Theta^w$  with the Thom space (see Section 2.1) of the unstable vector bundle  $\mathcal{U}_\Theta^w \rightarrow \mathcal{M}_\Theta^w$ . This relation is well known for hyperbolic linear flows.

**Theorem 7.4** *Assume the existence of a conformal reduction  $C_\phi$ . Then the Conley index of  $\mathcal{M}_\Theta^w$  is the homotopy class of the Thom space of the unstable vector bundle  $\mathcal{U}_\Theta^w \rightarrow \mathcal{M}_\Theta^w$ .*

**Proof:** We start the proof by defining

$$\mathbb{B}_1^u = \{\Psi(q \cdot v^u) : |q \cdot v^u| \leq 1\} \subset \mathbb{B}_1$$

and

$$\mathbb{B}_0^u = \{\Psi(q \cdot v^u) : |q \cdot v^u| = 1\} \subset \mathbb{B}_0.$$

We note that the inclusion  $i : \mathbb{B}_1^u \rightarrow \mathbb{B}_1$  and the deformation retract

$$r : \mathbb{B}_1 \rightarrow \mathbb{B}_1^u, \quad q \cdot v \mapsto q \cdot v^u,$$

are such that  $r \circ i = \text{id}_{\mathbb{B}_1^u}$  and, clearly, there exists a continuous homotopy  $h : [0, 1] \times \mathbb{B}_1 \rightarrow \mathbb{B}_1$  between the map  $i \circ r$  and the map  $\text{id}_{\mathbb{B}_1}$  such that

$$h([0, 1] \times \mathbb{B}_0) \subset \mathbb{B}_0.$$

Since the Conley index is given by the homotopy class of the quotient space  $\mathbb{B}_1/\mathbb{B}_0$  and the Thom space of the unstable vector bundle  $\mathcal{U}_{\mathcal{G}}^w$  is the quotient space  $\mathbb{B}_1^u/\mathbb{B}_0^u$ , we need to show that these quotient spaces are homotopically equivalent. Since  $i(\mathbb{B}_0^u) \subset \mathbb{B}_0$  and  $r(\mathbb{B}_0) = \mathbb{B}_0^u$ , we can define maps  $I$  and  $R$  such that the following diagrams commute

$$\begin{array}{ccccc} \mathbb{B}_1 & \xrightarrow{r} & \mathbb{B}_1^u & \xrightarrow{i} & \mathbb{B}_1 \\ p \downarrow & & p^u \downarrow & & \downarrow p \\ \mathbb{B}_1/\mathbb{B}_0 & \xrightarrow{R} & \mathbb{B}_1^u/\mathbb{B}_0^u & \xrightarrow{I} & \mathbb{B}_1/\mathbb{B}_0 \end{array} \quad (13)$$

where  $p$  and  $p^u$  are the respective projections. These diagrams imply that  $I$  and  $R$  are continuous maps and it is immediate to show that  $R \circ I = \text{id}_{\mathbb{B}_1^u/\mathbb{B}_0^u}$ . Finally, since

$$h([0, 1] \times \mathbb{B}_0) \subset \mathbb{B}_0,$$

we can define a map  $H$  such that the following diagram commutes

$$\begin{array}{ccc} [0, 1] \times \mathbb{B}_1 & \xrightarrow{h} & \mathbb{B}_1 \\ \text{id}_{[0,1]} \times p \downarrow & & \downarrow p \\ [0, 1] \times (\mathbb{B}_1/\mathbb{B}_0) & \xrightarrow{H} & \mathbb{B}_1/\mathbb{B}_0 \end{array} \quad (14)$$

showing that  $H$  is a continuous homotopy between the map  $I \circ R$  and the map  $\text{id}_{\mathbb{B}_1/\mathbb{B}_0}$  and therefore that  $\mathbb{B}_1/\mathbb{B}_0$  and  $\mathbb{B}_1^u/\mathbb{B}_0^u$  are homotopically equivalent spaces.  $\square$

**Corollary 7.5** *The following isomorphism in cohomology holds*

$$CH^{*+n_w}(\mathcal{M}_{\Theta}^w) \simeq H^*(\mathcal{M}_{\Theta}^w), \quad (15)$$

where  $n_w$  is the dimension of  $\mathcal{U}_{\Theta}^w$ . The cohomology coefficients are taken in  $\mathbb{Z}_2$  in the general case and in  $\mathbb{Z}$  if  $\mathcal{U}_{\Theta}^w$  is orientable.

**Proof:** Follows directly from Thom's isomorphism (see Section 2.1).  $\square$

Since there is also a Thom isomorphism in singular homology, the above isomorphisms also hold for the singular homology of the Conley index.

We now specialize the previous result to the situation of a contractible base (which is related to control flows).

**Proposition 7.6** *Assume the existence of a conformal reduction  $C_\phi$ . Put  $\Delta = \Theta(H)$  and take  $H_\Theta \in \text{cl}\mathfrak{a}^+$  such that  $\Theta = \Theta(H_\Theta)$ . If the base  $X$  is contractible, then the Conley index of  $\mathcal{M}_\Theta^w$  is the homotopy class of the Thom space of the vector bundle  $V_\Theta^+(H, w) \rightarrow \mathbb{F}(\Delta)_{H_0}$ , where  $H_0$  is the orthogonal projection of  $wH_\Theta$  in  $\mathfrak{a}(\Delta)$ . In particular, we have the following isomorphism in cohomology*

$$CH^{*+n_w}(\mathcal{M}_\Theta^w) \simeq H^*(\mathbb{F}(\Delta)_{H_0}),$$

where  $n_w$  is the dimension of  $V_\Theta^+(H, w)$ . The cohomology coefficients are taken in  $\mathbb{Z}_2$  in the general case and in  $\mathbb{Z}$  if  $V_\Theta^+(H, w)$  is orientable.

**Proof:** Since  $X$  is contractible  $C_\phi$  is trivial (see [24], Corollary 11.6) thus there exists a continuous global section  $\chi : X \rightarrow C_\phi$ . It follows that the application  $\lambda : X \times V_\Theta^+(H, w) \rightarrow \mathcal{U}_\Theta^w$ , given by

$$(x, v) \mapsto \frac{\chi(x) \cdot v}{|\chi(x) \cdot v|} \|v\|$$

is a homeomorphism whose inverse is given by

$$\chi(x) \cdot v \mapsto \left( x, \frac{v}{\|v\|} |\chi(x) \cdot v| \right),$$

where  $|\cdot|$  is the norm in  $\mathcal{U}_\Theta^w$  given by Proposition 7.1 and  $\|\cdot\|$  is some norm in  $V_\Theta^+(H, w)$ . Let

$$B_1^u = \{v \in V_\Theta^+(H, w) : \|v\| \leq 1\}$$

and

$$B_0^u = \{v \in V_\Theta^+(H, w) : \|v\| = 1\}.$$

It is then straightforward that  $\lambda(X \times B_1^u)$  is contained in the open set where  $\Psi$  is an homeomorphism and also that

$$\Psi(\lambda(X \times B_1^u)) = \mathbb{B}_1^u \quad \text{and} \quad \Psi(\lambda(X \times B_0^u)) = \mathbb{B}_0^u.$$

Thus we have

$$\mathbb{B}_1^u / \mathbb{B}_0^u \simeq (X \times B_1^u) / (X \times B_0^u)$$

whose homotopy class is, by Theorem 7.4, the Conley index of  $\mathcal{M}_\Theta^w$ . It remains to show that the space  $(X \times B_1^u) / (X \times B_0^u)$  and the space  $B_1^u / B_0^u$

are homotopically equivalent. We apply the same argument of Theorem 7.4 where the inclusion is given by

$$i : B_1^u \rightarrow X \times B_1^u, \quad v \mapsto (\bar{x}, v),$$

the deformation retract is given by

$$r : X \times B_1^u \rightarrow B_1^u, \quad (x, v) \mapsto v$$

and the homotopy is given by

$$h : [0, 1] \times (X \times B_1^u) \rightarrow X \times B_1^u, \quad (t, (x, v)) \mapsto (g(t, x), v)$$

where  $g : [0, 1] \times X \rightarrow X$  is some homotopy between  $\text{id}_X$  and the constant map  $\bar{x} \in X$ .  $\square$

Now we apply the abstract Morse equation to our flows. By Theorem 4.1, the finest Morse decomposition in  $\mathbb{E}_\Theta$  is given by  $\mathcal{M}_\Theta^w$ , for  $w \in \mathcal{W}_H \setminus \mathcal{W}/\mathcal{W}_\Theta$ . By Theorem 7.4 we have that  $CH^{*+n_w}(\mathcal{M}_\Theta^w) \simeq H^*(\mathcal{M}_\Theta^w)$ , where  $n_w$  is the dimension of the unstable bundle of  $\mathcal{M}^w$ . In the maximal flag bundle  $\mathbb{E}$ , item (ii) of Proposition 4.2 gives us that  $H^*(\mathcal{M}^w) \simeq H^*(\mathcal{M}^+)$ . It follows that

$$CP(t, \mathcal{M}^w) = \sum_{j \geq 0} t^{j+n_w} H^j(\mathcal{M}^w) = t^{n_w} P(t, \mathcal{M}^w) = t^{n_w} P(t, \mathcal{M}^+)$$

By Theorem 2.1 we have then immediately.

**Corollary 7.7** *Assume the existence of a conformal reduction  $C_\phi$  and suppose that the cohomology of the base  $X$  have finite rank in all dimensions. Then the Morse equation in the maximal flag bundle  $\mathbb{E}$  becomes*

$$P(t, \mathcal{M}^+) \sum_{w \in \mathcal{W}_H \setminus \mathcal{W}} t^{n_w} = P(t, \mathbb{E}) + (1+t)R(t). \quad (16)$$

*If the coefficient ring is the integers  $\mathbb{Z}$  then the coefficients of  $R(t)$  are non-negative.*

We note that where  $\mathcal{M}^+ \rightarrow X$  is a bundle with fiber  $\mathbb{F}(\Theta(\phi))$ , where  $\Theta(\phi)$  is the parabolic type of the flow  $\phi_t$ .

Equation (16) relates the topology of the attractor component and the unstable dimensions with the topology of the whole maximal flag bundle  $\mathbb{E}$ .

In the other flag bundles  $\mathbb{E}_\Theta$  the Morse components  $\mathcal{M}_\Theta^w$  are not necessarily homeomorphic, so that the Morse equation is not simple as above. Nevertheless, if the flow is regular, then each Morse component  $\mathcal{M}_\Theta^w$  in  $\mathbb{E}_\Theta$  is homeomorphic to the base space  $X$  so that the Morse equation for the flow in  $\mathbb{E}_\Theta$  becomes

$$P(t, X) = \sum \{t^{n_w} : w \in \mathcal{W}/\mathcal{W}_\Theta\} = P(t, \mathbb{E}_\Theta) + (1+t)R(t).$$

Note that in the maximal flag bundle we have  $\mathcal{W}_\Theta = \{1\}$ .

For the attractor component we can drop the assumption on the existence of a conformal reduction.

**Corollary 7.8** *The cohomological Conley index of the attractor in  $\mathbb{E}_\Theta$  is given by*

$$CH^*(\mathcal{M}_\Theta^+) = H^*(\mathcal{M}_\Theta^+),$$

for every  $\Theta \subset \Sigma$ . Furthermore, if  $\Theta \subset \Theta(\phi)$  or  $\Theta(\phi) \subset \Theta$ , then the Conley index of  $\mathcal{M}_\Theta^+$  is the homotopy class of the flag bundle  $\mathcal{M}_\Theta^+ \rightarrow X$ .

Put  $\Delta = \Theta(H)$  and take  $H_\Theta \in \text{cl}^+$  such that  $\Theta = \Theta(H_\Theta)$ . If the base  $X$  is contractible then the cohomological Conley index of the attractor in  $\mathbb{E}_\Theta$  is given by

$$CH^*(\mathcal{M}_\Theta^+) = H^*(\mathbb{F}(\Delta)_{H_0}),$$

where  $H_0$  is the orthogonal projection of  $H_\Theta$  in  $\mathfrak{a}(\Delta)$ . Also, if  $\Theta \subset \Theta(\phi)$  or  $\Theta(\phi) \subset \Theta$ , then the Conley index of  $\mathcal{M}_\Theta^+$  is the homotopy class of the flag manifold  $\mathbb{F}(\Delta)_{H_0}$ .

**Proof:** The first assertion follows directly from Corollary 5 of [20]. The second assertion follows from Corollary 7.2, proceeding as in the proof of Theorem 7.4. The third and fourth assertions follow from the first and second assertions, item (i) of Proposition 4.2 and Corollary 11.6 of [24], since  $X$  is contractible.  $\square$

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